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The geometry of the hermitian matrix model and lattices for the NLS and dNLS hierarchies *

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Abstract

The geometrical description of the Nonlinear Schrödinger–Toda system hierarchy in the Sato Grassmannian with the action of the translation group is applied to the Hermitian one-matrix model. A family of derivative Nonlinear Schrödinger system hierarchies with its lattices—associated with the Volterra chain—which are auto-Bäcklund transformations, is analyzed from a geometrical point of view. The Sato periodic flag manifold with the line bundles over it turns out to be the proper infinite-dimensional manifold in this case. The lattice appears as a square root of the action of the translation group; this can be understood as a reduction of the action of a translation group of a larger loop group. The reduction $t_{2n+1} = 0$ of the Hermitian one-matrix model, essential in the double scaling limit, is shown to be described in terms of the derivative Nonlinear Schrödinger–Volterra system hierarchy. The role of the heat hierarchy, self-similarity and auto-Bäcklund transformations is pointed out. A characterization in Sato's Grassmannian and periodic flag manifold of the Hermitian one-matrix model is given. In the latter case we are concerned with the $t_{2n+1} = 0$ reduction.

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1. Introduction

Since the seminal papers [4], in which the role played by the Korteweg–de Vries equation in two-dimensional quantum gravity through a double scaling limit of the

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Hermitian one-matrix model was pointed out, there have been a remarkable number of papers analyzing different aspects of the theory. We must also recall the important contributions of Witten [29] and Kontsevich [16] regarding the description of the moduli space for the intersection theory of complex curves with the use of the Korteweg-de Vries hierarchy.

In particular, the geometry associated with these systems has been studied in a number of papers [15,20,11] for different matrix models. The Sato Grassmannian is the infinite-dimensional manifold normally used; however, the Sato periodic flag manifold is necessary as well [20].

Integrability aspects of matrix models appear already before the double scaling limit is taken, see [9] and references therein. For the Hermitian one-matrix model, the semi-infinite Toda chain hierarchy and a string equation model the partition function. For the double scaling limit of the Hermitian one-matrix model one requires the odd sources to vanish; in that case the Volterra hierarchy appears.

Recently [3] the Non-linear Schrödinger hierarchy has appeared in connection with the Hermitian one-matrix model before the double scaling limit is taken. This was achieved within the pseudo-differential formalism for the Kandomtsev–Petviashvilii hierarchy. Moreover, this technique has been extended to study multi-matrix models. The string equation is a Galilean self-similar condition [12]. However, the Toda chain models auto-Bäcklund transformations of the Non-linear Schrödinger hierarchy, as has been well known for years already [27]. Therefore, the connection of this integrable hierarchy with the Hermitian one-matrix model is obvious once the role of the Toda chain is discovered.

On the other hand the study of lattices and auto-Bäcklund transformations of integrable hierarchies is currently a subject of interest [24].

Observe that the Non-linear Schrödinger system hierarchy can be described as flows in the Sato Grassmannian $\mathrm{Gr}^{(2)}$, and the Toda chain reflects the action of the translation group of the loop group $L\mathrm{SL}(2,\mathbb{C})$—an Abelian subgroup of the affine Weyl group—[2,26,10]. Thus, it is natural to apply such a description to the Hermitian one-matrix model.

This paper is mainly dedicated to the study of the geometry of the Hermitian one-matrix model. However, we are interested in lattices for derivative Non-linear Schrödinger system type hierarchies and its geometrical description. Firstly we analyze the non-reduced Hermitian one-matrix model and then the reduced case.

The second section is an introduction of well known facts regarding the Hermitian one-matrix model and its relation with the Toda hierarchy. We introduce also the Non-linear Schrödinger hierarchy and its auto-Bäcklund transformations given by the Toda chain. The next section, Section 3, is dedicated to the local symmetries of the Non-linear Schrödinger–Toda hierarchy, translations and scaling and Galilean transformations. A generalized string equation is presented and we prove Proposition 7, appearing in [12] without proof, where the Galilean self-similarity is shown to imply weighted scaling self-similarity; in [12] a detailed study of the geometrical aspects of self-similarity is given. Proposition 8 gives a integrable system characterization of the N-dimensional Hermitian one-matrix model in terms of the heat hierarchy and auto-Bäcklund transformations. We end the section by stating how the general string equation evolves with the auto-Bäcklund transformations given by the Toda chain. The Grassmannian description, via the Birkhoff factorization in the loop group $L\mathrm{SL}(2,\mathbb{C})$, of the Non-linear Schrödinger–Toda hierarchy is introduced in Section 4 and there one can find explicitly with which points the N-dimensional Hermitian one-matrix model is associated in the Sato Grassmannian.

The next section deals with a one-parameter family of derivative Non-linear Schrödinger type hierarchies, its lattices and the $t_{2n+1} = 0$ reduction of the Hermitian one-matrix model necessary in order to perform the double scaling limit. Section 5 is devoted to the introduction of these integrable hierarchies and of the associated lattices. The lattices will be constructed in Section 6 by geometrical means and are shown to give local and non-local auto-Bäcklund transformations of the integrable hierarchy. In this section these integrable systems are described with the use of the periodic flag manifold and line bundles over it, via the principal subgroup $L(\mathrm{SL}(2,\mathbb{C}),C)$ of $L\mathrm{SL}(2,\mathbb{C})$ and a factorization problem given by a classical r-matrix which is not a difference of projectors. Then we construct a *square root* of the action translation group of $L\mathrm{SL}(2,\mathbb{C})$ that reduces to the principal subgroup and gives the lattices mentioned. The points in the Sato periodic flag manifold corresponding to the $t_{2n+1} = 0$ reduction of the N-dimensional Hermitian one-matrix model are given at the end of this section. Finally, in Section 7 a Miura type map between the Non-linear Schrödinger equation and its derivative deformation is introduced. The geometrical description in terms of fibrations is given and we find a *square root* of the Toda chain. We end by showing that the *square root* can be obtained as a reduction of the action of the translation groups of larger loop groups, for example $L\mathrm{SL}(3,\mathbb{C})$, giving generalized Non-linear Schrödinger hierarchies.

2. The Hermitian one-matrix model and the NLS–Toda chain hierarchy

This section is a schematic introduction of the Hermitian one-matrix model (HMM) and its relation with the Non-linear Schrödinger–Toda chain hierarchy. Firstly, we remind the reader of the standard construction that connects the HMM with the Toda chain hierarchy. Then, the Non-linear Schrödinger (NLS) hierarchy and its auto-Bäcklund transformations are introduced.

2.1. The HMM and the Toda chain

The HMM [9] has as partition function

$$Z_N(\boldsymbol{t}) := c_N \int \mathrm{d}M \exp(\operatorname{Tr} V(\boldsymbol{t},M)),$$

where $\boldsymbol{t} := \{t_n\}_{n\geq 0}$, $t_n < 0$, are the sources or couplings of the model, M is a N by N Hermitian matrix, i.e. $M = M^\dagger$, c_N is a normalization constant and

$$V(t,\lambda) := \sum_{n\geq 0} \lambda^n t_n.$$

Consider the scalar product

$$\langle f, g\rangle := \int_{\mathbb{R}} \mathrm{d}\lambda \exp(V(\lambda)) f(\lambda) g(\lambda),$$

and the polynomials $\{P_n\}_{n\geq 0}$ of the form

$$P_n(\lambda) = \lambda^n + \mathrm{O}(\lambda^{n-1}),$$

that fulfill the relations

$$\langle P_n, P_m\rangle = \delta_{nm} \exp(\phi_n).$$

Then, the partition function can be expressed as follows:

$$Z_N(t) = \exp\left(\sum_{n=0}^{N-1} \phi_n(t)\right).$$

The dynamics of ϕ_n in t gives the action of the renormalization group in the partition function. To analyze the dependence of the ϕ_n in the couplings t one introduces the semi-infinite matrices $\mathcal{Q}, \mathcal{P}$ defined as

$$\exp(\phi_n)\mathcal{Q}_{nm} = \langle P_n, \lambda P_m\rangle, \qquad \exp(\phi_n)\mathcal{P}_{nm} = \langle P_n, (\mathrm{d}/\mathrm{d}\lambda) P_m\rangle.$$

The explicit form of $\mathcal{Q}$ follows from the recurrence relation for the orthogonal polynomials:

$$\lambda P_n(\lambda) = P_{n+1}(\lambda) + S_n P_n(\lambda) + R_n P_{n-1}(\lambda),$$

with $R_0 = 0$ and

$$R_{n+1} = \exp(\phi_{n+1} - \phi_n), \qquad S_n = \partial\phi_n,$$

where $\partial = \partial/\partial t_1$. It turns out that $\mathcal{Q}$ is a tridiagonal Jacobi matrix of the form

$$\mathcal{Q} = \sum_{n\geq 0} (E_{n,n+1} + S_n E_{n,n} + R_{n+1} E_{n+1,n}),$$

where $E_{n,m}$ is the matrix with its non-zero entry, which is 1, located at the site given by the intersection of the nth row with the mth column. Any semi-infinite matrix M splits into its strictly upper triangular part M_+ and its lower triangular part M_-, i.e. $M = M_+ + M_-$. It easily shown that

$$\mathcal{P} = -\left(\frac{\mathrm{d}V}{\mathrm{d}\lambda}(\mathcal{Q})\right)_+.$$

If $\mathrm{d} = \sum_{n\geq 0} \partial_n \mathrm{d}t_n$, where $\partial_n = \partial/\partial t_n$, is the exterior derivative associated to the couplings $\boldsymbol{t}$ and we construct the 1-form

$$\omega := \sum_{n\geq 0} \mathrm{d}t_n \mathcal{Q}^n,$$

then $\mathcal{Q}$ satisfies

$$\mathrm{d}\mathcal{Q} = [\omega_+, \mathcal{Q}],$$

and the constraint, the so called string equation

$$[\mathcal{P}, \mathcal{Q}] = 1.$$

The flow equations are those of the semi-infinite Toda chain hierarchy, in particular

$$\partial R_n = R_n(S_n - S_{n-1}), \qquad \partial S_n = R_{n+1} - R_n,$$

or

$$\partial^2 \phi_n = \exp(\phi_{n+1} - \phi_n) - \exp(\phi_n - \phi_{n-1}),$$

the well-known semi-infinite Toda chain equation.

2.2. The NLS system hierarchy and the Toda chain

Recently, Bonora and Xiong [3] pointed out the role of the Non-Linear Schrödinger (NLS) system hierarchy. Their analysis is based on KP type pseudo-differential operators. In Integrable Systems theory the connection between the NLS system hierarchy and the semi-infinite Toda chain has been known for years already [13,2]. In fact the Toda chain appears in the NLS system modeling auto-Bäcklund transformations. In a series of papers [24] one can find a systematic study, based on symmetries, of lattice-differential integrable systems.

The functions defined by

$$p^{(n)} := R_n \exp(\partial^{-1} S_{n-1}) = \exp(\phi_n),$$
$$q^{(n)} := -\exp(-\partial^{-1} S_{n-1}) = -\exp(-\phi_{n-1})$$

for each n satisfy the NLS system hierarchy that we are about to introduce.

Definition 1. The NLS system hierarchy for p, q is the following collection of compatible equations:

$$\partial_n p = p_{n+1}, \qquad \partial_n q = -q_{n+1},$$

where $n \geq 0$ and p_n, q_n and h_n are defined recursively by the relations

$$p_n = \partial p_{n-1} + p h_{n-1},$$
$$q_n = -\partial q_{n-1} + q h_{n-1},$$
$$\partial h_n = 2(p q_n - q p_n), \quad n \geq 1$$

with the initial data

$$p_0 = q_0 = 0, \qquad h_0 = 1.$$

For $n = 2$ the equations are those of the NLS system

$$\partial_2 p = \partial^2 p - 2p^2 q, \qquad \partial_2 q = -\partial^2 q + 2pq^2.$$

The Toda chain hierarchy is complemented by the Toda chain equation, which in the new variables reads

$$q^{(n+1)} = -1/p^{(n)}, \qquad p^{(n+1)} = p^{(n)}\left(-p^{(n)} q^{(n)} + \partial^2 \ln p^{(n)}\right).$$

For a solution of the NLS at the site n these equations give a new solution at the site $n+1$, an auto-Bäcklund transformation.

Observe that $\partial \ln Z_N = \sum_{n=0}^{N-1} S_n$ so that, using the Toda chain equations, one has for the specific heat of the HMM [3]

$$\partial^2 \ln Z_N = \sum_{n=0}^{N-1} (R_{n+1} - R_n) = R_N = -p^{(N)} q^{(N)}.$$

3. The string equation and self-similarity

As we have seen the renormalization flows in the HMM are the integrable flows of the semi-infinite Toda chain hierarchy, which happens to be equivalent to the coupling of the NLS system hierarchy with the semi-infinite Toda chain equation, the latter modeling the auto-Bäcklund transformations of the former. However, there is a constraint, the string equation, that must be satisfied. As was shown in [3] the string equation can be formulated in each site n in the lattice in terms of the functions R_n, S_n and therefore in terms of $p^{(n)}, q^{(n)}$. Later it was proven that this constraint is the Galilean self-similarity condition for the solutions $p^{(n)}, q^{(n)}$ of the NLS system hierarchy [12]. For the relation with the Heisenberg ferromagnetic model see [18].

Now, we introduce the translations and the Galilean and scaling transformations, local symmetries of the NLS system hierarchy and compatible with the discrete symmetry given by the Toda chain.

Definition 2. Let ϑ,

$$\vartheta(\boldsymbol{t}) := \boldsymbol{t} + \boldsymbol{\theta},$$

be the action of translations, where

$$\boldsymbol{\theta} := \{\theta_n\}_{n\geq 0} \in \mathbb{C}^\infty.$$

Then, the following proposition follows:

Proposition 3. *If (p,q) is a solution to the NLS system hierarchy then so is $(\boldsymbol{\vartheta}^* p, \boldsymbol{\vartheta}^* q)$.*

However, there are also two local non-isospectral symmetries. One is the scaling symmetry, and the other is the Galilean symmetry. Now, we define them.

Definition 4. The Galilean transformation $\boldsymbol{t} \mapsto \gamma_a(\boldsymbol{t})$ is given by

$$\gamma_a(\boldsymbol{t})_n := \sum_{m\geq 0} \binom{n+m}{m} a^m t_{n+m},$$

where $a \in \mathbb{C}$.

The scaling transformation $\boldsymbol{t} \mapsto \varsigma_b(\boldsymbol{t})$ is represented by the relations

$$\varsigma_b(\boldsymbol{t})_n := e^{nb} t_n,$$

where $b \in \mathbb{C}$.

One can show that

Proposition 5. *If (p,q) is a solution of the NLS system hierarchy then so are $(\gamma_a^* p, \gamma_a^* q)$ and $(e^b \varsigma_b^* p, e^b \varsigma_b^* q)$.*

The related fundamental vector fields, infinitesimal generators of the action of translation, Galilean and scaling transformations are given by

$$\partial_n, \quad n \geq 0, \qquad \boldsymbol{\gamma} = \sum_{n\geq 0} (n+1) t_{n+1} \partial_n, \qquad \boldsymbol{\varsigma} = \sum_{n\geq 1} n t_n \partial_n,$$

respectively. They generate the linear space $\mathbb{C}\{\partial_n, \boldsymbol{\varsigma}, \boldsymbol{\gamma}\}_{n\geq 0}$, which is the Lie algebra of local symmetries of the NLS system hierarchy; the non-trivial Lie brackets are

$$[\partial_n, \boldsymbol{\varsigma}] = n\partial_n, \quad [\partial_{n+1}, \boldsymbol{\gamma}] = (n+1)\partial_n, \quad [\boldsymbol{\varsigma}, \boldsymbol{\gamma}] = 2\boldsymbol{\gamma}.$$

Consider the following vector field belonging to this Lie algebra,

$$X := \boldsymbol{\vartheta} + a\boldsymbol{\gamma} + b\boldsymbol{\varsigma}, \quad \boldsymbol{\vartheta} = \sum_{n\geq 0} \theta_n \partial_n,$$

defining a superposition of translations, Galilean and scaling transformations.

If (p,q) is a solution of the NLS system hierarchy then there is a 1-parameter family of solutions (p_τ, q_τ) generated by the vector field X. A self-similar solution under any of the mentioned symmetries is a solution which remains invariant under the corresponding transformation.

Then we have,

Proposition 6. *A solution (p,q) of the NLS system hierarchy is self-similar under the action of the vector field X if and only if it satisfies the generalized string equations*

$$Xp + bp = 0, \qquad Xq + bq = 0, \tag{3.1}$$

The Galilean self-similarity condition is the string equation for the HMM.

Notice that when $X = \gamma + \sum_{n\geq 0} \theta_n \partial_n$, one can perform the coordinate transformation $t_{n+1} \mapsto t_{n+1} + \theta_n/(n+1)$. Thus, the coefficient θ_n is equivalent to a shift in the time coordinate t_{n+1}.

Now, if $X = \varsigma + \sum_{n\geq 0} \theta_n \partial_n$, we can define the transformation $t_{n+1} \mapsto t_{n+1} + \theta_{n+1}/(b(n+1))$ and obtain in the new coordinates a vector field corresponding to scaling and a term of the type $\theta_0\partial_0$. This last term can be understood as follows. Given a solution (p,q) to the NLS system hierarchy then $(\exp(b(1+2\theta_0))\varsigma_b^* p, (\exp(b(1-2\theta_0))\varsigma_b^* q)$ is a solution as well. So solutions self-similar under the vector field X correspond, in adequate coordinates, to self-similarity under this particular scaling, which we shall call $(1+2\theta_0, 1-2\theta_0)$ weighted scaling.

Now we shall prove that Galilean self-similarity implies scaling self-similarity. We have

Proposition 7. *If (p,q) is a solution to the NLS system hierarchy self-similar under the action of the vector field*

$$\gamma + \sum_{n\geq 0} \theta_n \partial_n,$$

then it is also self-similar under the action of the vector field

$$\varsigma + \sum_{n\geq 0} \theta_n \partial_{n+1} - \left(\sum_{n\geq 1} \theta_n h_{n+1}|_{t=0} \right) \partial_0.$$

This proposition simply says that the L_{-1}-Virasoro constraint implies the L_0-Virasoro constraint.

Proof. We have

$$\left(\gamma + \sum_{n\geq 0} \theta_n \partial_n \right) p = \left(\gamma + \sum_{n\geq 0} \theta_n \partial_n \right) q = 0.$$

Therefore, we obtain the relations

$$\left(\gamma + \sum_{n\geq 0} \theta_n \partial_n \right) p_{n+1} = -np_n, \qquad \left(\gamma + \sum_{n\geq 0} \theta_n \partial_n \right) q_{n+1} = -nq_n,$$

where, for example, we have used the fact that $2p_{n+1} = \partial_n p$, p is killed by $\boldsymbol{\gamma}+\sum_{n\geq 0}\theta_n\partial_n$ and the commutation relation of this vector field and ∂_n. One can equally deduce

$$\left(\boldsymbol{\gamma}+\sum_{n\geq 0}\theta_n\partial_n\right) h_{n+1} = -nh_n.$$

Because

$$\partial_{n+1}p = (\tfrac{1}{2}\partial_n + 2h_{n+1})p\,, \qquad \partial_{n+1}q = -(\tfrac{1}{2}\partial_n + 2h_{n+1})q$$

it follows that

$$\left(\boldsymbol{s}+\sum_{n\geq 0}\theta_n\partial_{n+1}\right) p = \frac{1}{2}\left(\boldsymbol{\gamma}+\sum_{n\geq 0}\theta_n\partial_n\right) p + 2\left(\sum_{n\geq 1}(nt_n+\theta_{n-1})h_n\right) p\,,$$
$$\left(\boldsymbol{s}+\sum_{n\geq 0}\theta_n\partial_{n+1}\right) q = -\frac{1}{2}\left(\boldsymbol{\gamma}+\sum_{n\geq 0}\theta_n\partial_n\right) q - 2\left(\sum_{n\geq 1}(nt_n+\theta_{n-1})h_n\right) q.$$

Observe that

$$\partial_n \sum_{m\geq 1}(mt_m+\theta_{m-1})h_m = nh_n + \left(\boldsymbol{\gamma}+\sum_{m\geq 0}\theta_m\partial_m\right) h_{n+1}.$$

Hence, when (p,q) is self-similar under $\boldsymbol{\gamma}+\sum_{m\geq 0}\theta_m\partial_m$ we have

$$\sum_{n\geq 1}(nt_n+\theta_{n-1})h_n = \sum_{n\geq 0}\theta_n h_{n+1}|_{t=0}\,.$$

This implies

$$\left(\boldsymbol{s}+\sum_{n\geq 0}\theta_n\partial_{n+1}\right) p - 2\left(\sum_{n\geq 0}\theta_n h_{n+1}|_{t=0}\right) p = 0\,,$$
$$\left(\boldsymbol{s}+\sum_{n\geq 0}\theta_n\partial_{n+1}\right) q + 2\left(\sum_{n\geq 0}\theta_n h_{n+1}|_{t=0}\right) q = 0,$$

and the proposition follows. □

Observe that the NLS system hierarchy contains as reduction the heat hierarchy. In fact when $q=0$ then p satisfies the heat hierarchy,

$$\partial_n p = \partial^n p.$$

The general solution to it will be

$$p(t) = \int \mathrm{d}\lambda\; P(\lambda)\exp(V(t,\lambda))\,; \tag{3.2}$$

however, if we want this solution to be Galilean self-similar we need $\gamma p = 0$, thus—integrating by parts—$\int \mathrm{d}\lambda\; dP(\lambda)/\mathrm{d}\lambda \exp(V(t,\lambda)) = 0$, so that P is constant almost everywhere. The starting point $p^{(1)} = \exp(\phi_1)$, $q^{(1)} = -\exp(-\phi_0)$, is the result of applying the Bäcklund transformation to $p^{(0)} = \exp(\phi_0)$, $q^{(0)} = 0$, which is a heat hierarchy reduction with $p^{(0)}(t) = \int \mathrm{d}\lambda \exp(V(t,\lambda))$.

Proposition 8. *The N-dimensional HMM is obtained after N consecutive auto-Bäcklund transformations of the solution of the NLS system hierarchy which is a Galilean self-similar solution of the heat hierarchy.*

Obviously, the auto-Bäcklund transformations of the NLS system hierarchy given by the Toda chain commute with the translational symmetries. Moreover, they commute also with the non-isospectral Galilean and scaling transformations. The following proposition will be proven in the next section by geometrical means.

Proposition 9. *Given a solution, say $p^{(n)}, q^{(n)}$, of the NLS system hierarchy, self-similar under the action of $X^{(n)} = \boldsymbol{\vartheta} + a\boldsymbol{\gamma} + b\boldsymbol{\varsigma}$, then its Bäcklund transform, given by the Toda chain equations, say $p^{(n+1)}, q^{(n+1)}$, is self-similar under the action of the vector field $X^{(n+1)} = X^{(n)} + b\partial_0$.*

Observe that this result implies that the weights in the weighted scaling case are shifted by a Toda chain Bäcklund transformation.

4. The Grassmannian and the NLS–Toda hierarchy

The NLS–Toda system hierarchy, as was shown in [2], has a geometrical interpretation in an infinite-dimensional Grassmannian [21]. This can be realized once a Birkhoff factorization problem is considered.

We define the vacuum wave function, an infinite set of commuting flows in the loop group [21] $L\mathrm{SL}(2,\mathbb{C})$ of smooth maps from the circle $S^1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ to the simple Lie group $\mathrm{SL}(2,\mathbb{C})$ of 2×2 unity determinant matrices as follows:

$$\psi(t,\lambda) := \exp(V(t,\lambda)\; H/2)\cdot g(\lambda)\,, \tag{4.1}$$

where g is the initial condition and we are using the standard Weyl basis $\{E, H, F\}$ of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$.

The loop algebra $L\mathfrak{sl}(2,\mathbb{C})$ when extended to the affine Lie algebra of type $A_1^{(1)}$ has an associated affine Weyl group generated by the reflections r_0, r_1 defined by the simple roots α_0, α_1 [14]. The translation group generated by $T = r_1 r_0$ is an Abelian subgroup of it. Observe that if $\delta = \alpha_0 + \alpha_1$ is the imaginary root then the $A_1^{(1)}$-root system is $\Delta = \{n\delta, n\delta \pm \alpha_1\}_{n\in\mathbb{Z}}$, and the action of T is given by $T(n\delta) = n\delta$ and $T(n\delta \pm \alpha_1) = (n \mp 2)\delta \pm \alpha_1$. At the level of the loop algebra the adjoint action of

$$T^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},$$

is an action of the translation group. This element commutes with H.

Definition 10. The shifted vacuum wave functions are defined by

$$\psi^{(n)} := T^{-n} \cdot \psi.$$

Observe that

$$\psi^{(n)}(t, \lambda) := \exp(V(t,\lambda)H/2) \cdot g^{(n)}(\lambda),$$

where $g^{(n)} := T^{-n} \cdot g$. Denote by $L^+\mathrm{SL}(2,\mathbb{C})$ those loops which have a holomorphic extension to the interior of S^1 [21] and by $L_1^-\mathrm{SL}(2,\mathbb{C})$ those which extend analytically to the exterior of the circle and are normalized by the identity at ∞.

The Birkhoff factorization problem for a given $\psi^{(n)}(t)$ consists in finding the representation

$$\psi^{(n)} = \left(\psi_-^{(n)}\right)^{-1} \cdot \psi_+^{(n)}, \tag{4.2}$$

where $\psi_-^{(n)}(t) \in L_1^-\mathrm{SL}(2,\mathbb{C})$ and $\psi_+^{(n)}(t) \in L^+\mathrm{SL}(2,\mathbb{C})$, and is connected with the NLS system hierarchy. The element $\psi_-^{(n)}$ can be parametrized by functions $p^{(n)}, q^{(n)}$ in such a way that $\psi_-^{(n)}$ is a solution to the factorization problem if and only if $p^{(n)}, q^{(n)}$ is a solution to the NLS system hierarchy [2,10]. To this end, one considers the equation that follows from Eq. (4.2),

$$\partial\psi_-^{(n)} \cdot \left(\psi_-^{(n))}\right)^{-1} = -P_- \operatorname{Ad}\psi_-^{(n)} \lambda\ H/2,$$

and factorizes $\psi_-^{(n)}$ as follows:

$$\psi_-^{(n)} = \zeta^{(n)} \cdot \phi^{(n)},$$

where

$$\ln \zeta^{(n)} = \sum_{m\geq 1} \lambda^{-m} Z_m^{(n)}, \quad \phi^{(n)} = \exp\left(\sum_{m\geq 1} \lambda^{-m} \Phi_m^{(n)} H\right);$$

now $Z_m^{(n)}(t) \in \operatorname{Im}\operatorname{ad} H$ and $\partial_j \Phi_m^{(n)}$ can be expressed as polynomials in $p^{(n)}, q^{(n)}$ and its ∂-derivatives.

From Eq. (4.2) it also follows that

$$\chi^{(n)} := \mathrm{d}\psi_+^{(n)} \cdot \left(\psi_+^{(n)}\right)^{-1} = P_+ \operatorname{Ad}\psi_-^{(n)} \left(\mathrm{d}V\, H/2\right). \tag{4.3}$$

Here $\mathrm{id} = P_+ + P_-$ is the resolution of the identity related to the splitting

$$L\mathfrak{sl}(2,\mathbb{C}) = L^+\mathfrak{sl}(2,\mathbb{C}) \oplus L_1^-\mathfrak{sl}(2,\mathbb{C}).$$

Then, $\chi^{(n)}$ is the zero-curvature 1-form for the NLS system hierarchy, therefore the integrable hierarchy is equivalent to the condition

$$\left[\mathrm{d}-\chi^{(n)},\mathrm{d}-\chi^{(n)}\right]=0. \tag{4.4}$$

One also has

$$\mathcal{T}_+^{(n)} := \psi_+^{(n+1)}\cdot\left(\psi_+^{(n)}\right)^{-1} = \psi_-^{(n+1)}\cdot\mathcal{T}^{-1}\cdot\left(\psi_-^{(n)}\right)^{-1}, \tag{4.5}$$

and the lattice equations—Toda chain hierarchy—are

$$\chi^{(n+1)} = \mathrm{d}\mathcal{T}_+^{(n)}\cdot\left(\mathcal{T}_+^{(n)}\right)^{-1} + \mathrm{Ad}\,\mathcal{T}_+^{(n)}\chi^{(n)}. \tag{4.6}$$

The parametrization of $\psi_-^{(n)}$ in terms of $p^{(n)},q^{(n)}$ gives

$$\chi^{(n)} = \sum_{m\geq 0} L_m^{(n)}dt_m, \tag{4.7}$$

where

$$L_m^{(n)}(\lambda) := \sum_{j=0}^{m}\lambda^j Q_{m-j}^{(n)}, \quad Q_m^{(n)} := p_m^{(n)}E + h_m^{(n)}H + q_m^{(n)}F,$$

and

$$\mathcal{T}_+^{(n)} = \begin{pmatrix} \lambda-\partial\ln p^{(n)} & p^{(n)} \\ -(p^{(n)})^{-1} & 0 \end{pmatrix}.$$

The equation corresponding to t_1 is the semi-infinite Toda chain. The system of equations (4.4) and (4.6) is the NLS–Toda system hierarchy.

Observe that a solution to the NLS system hierarchy is fixed by the coset $g\cdot L^+\mathrm{SL}(2,\mathbb{C})$. However, the homogeneous space $L\mathrm{SL}(2,\mathbb{C})/L^+\mathrm{SL}(2,\mathbb{C})$ is isomorphic to the Grassmannian $\mathrm{Gr}^{(2)}$ [21]. Sato's Grassmannian [22] is the set of subspaces, say W, of $\mathcal{H}=\mathbb{C}^2[[\lambda^{-1},\lambda]$, the Laurent expansions, *commensurable* with $\mathcal{H}_+=\mathbb{C}^2[\lambda]$, the Taylor expansions, and such that $\lambda W\subset W$. In the Segal–Wilson version $\mathcal{H}$ is replaced by the Hilbert space $L^2(S^1,\mathbb{C}^2)$.

The Baker function $w(t)\in L\mathrm{SL}(2,\mathbb{C})$ corresponds to [28,6]

$$w=\psi_-\cdot\exp(tH)=\psi_+\cdot g^{-1}.$$

If we introduce the notation

$$g=\begin{pmatrix}\varphi_1 & \tilde\varphi_1\\ \varphi_2 & \tilde\varphi_2\end{pmatrix},$$

then we have the associated subspace

$$W=\mathbb{C}\left\{\lambda^n\left(\tilde\varphi_2,\ -\tilde\varphi_1\right),\lambda^n\left(\varphi_2,\ -\varphi_1\right)\right\}_{n\geq 0},$$

with $\lambda W \subset W$, in the Grassmannian $\mathrm{Gr}^{(2)}$, [21,23]. The Baker function is the unique function with its rows taking its values in W such that $P_+(w\cdot\exp(-tH)) = 1$. Obviously we have

$$\partial_1 w = L_1 w$$

and also

$$\partial_n w = L_n w.$$

The rows of the adjoint Baker function $w^* = (w^{-1})^t$ are maps into the subspace

$$W^* = \mathbb{C}\left\{\lambda^n \Phi, \lambda^n \tilde{\Phi}\right\}_{n\geq 0} \in \mathrm{Gr}^{(2)},$$

where

$$\Phi := (\varphi_1, \ \varphi_2), \qquad \tilde{\Phi} := (\tilde{\varphi}_1, \ \tilde{\varphi}_2).$$

We shall adopt this subspace as a representative of the coset $g \cdot L^+\mathrm{SL}(2,\mathbb{C})$.

The discrete map between initial conditions $g^{(n)} \mapsto g^{(n+1)}$ models the auto-Bäcklund transformations for the NLS system. For the Baker function on has

$$w^{(n)} \mapsto w^{(n+1)} = \mathcal{T}_+^{(n)} w^{(n)} T.$$

It was shown in [12] that those initial conditions g giving self-similar solutions under the action of the vector field X are characterized by

$$\delta g \cdot g^{-1} + \mathrm{Ad}\, g\, K = (\theta + f)H, \tag{4.8}$$

where

$$\delta := (a + b\lambda)\, \mathrm{d}/\mathrm{d}\lambda,$$

for some $K \in L^+\mathfrak{sl}(2,\mathbb{C})$ and some $f \in L_1^-\mathbb{C}$, where $\theta(\lambda) := \sum_{n\geq 0} \theta_n \lambda^n$. From this condition Proposition 9 follows.

Proof of Proposition 9. Given a solution $g^{(n)}$ with $f^{(n)} \in L_1^-\mathbb{C}$ then $g^{(n+1)} = T^{-1} \cdot g^{(n)}$. Observing that $\delta T^{-1} \cdot T = (b + a\lambda^{-1})H$ and $\mathrm{Ad}\, T^{-1} H = H$ one concludes that $g^{(n+1)}$ satisfies (4.8) with $f^{(n+1)} = f^{(n)} + a\lambda^{-1} \in L_1^-\mathbb{C}$ and $\theta^{(n+1)} = \theta^{(n)} + b$. □

The self-similar solutions under the action of the vector field X of the heat hierarchy are as stated in the following

Proposition 11. *The function*

$$p(\mathbf{t}) = \int_{\mathbb{R}} d\lambda\, \exp(V(\mathbf{t},\lambda) + \tilde{\theta}(\lambda)),$$

where

$$\tilde{\theta}(\lambda) := \int^{\lambda} d\mu \frac{\theta(\mu)}{a+b\mu},$$

is a solution of the heat hierarchy invariant under the action of the vector field X.

Proof. The generalized string equation (3.1) when applied to $q=0$ and p given by Eq. (3.2) implies

$$\frac{\mathrm{d}P}{\mathrm{d}\lambda}(\lambda) = \frac{\theta(\lambda)}{a+b\lambda} P(\lambda)$$

almost everywhere, from which the proposition follows. □

Notice that when $a=b=0$ the definition of $\tilde{\theta}$ fails, only constants are translational self-similar solutions of the heat hierarchy. The heat hierarchy reduction is equivalent to the following nilpotency property of the initial condition:

$$g(\lambda) = \begin{pmatrix} 1 & f(\lambda) \\ 0 & 1 \end{pmatrix}.$$

From the results of [12] one can deduce that the self-similar solution p of the heat hierarchy is connected with the unique solution P of the following ODE:

$$(a+b\lambda)\frac{\mathrm{d}f}{\mathrm{d}\lambda}(\lambda) - 2\theta(\lambda)\ f(\lambda) + \sum_{n\geq 0} \theta_n \sum_{m=0}^{n-1} \lambda^m \int_{\mathbb{R}} \mathrm{d}\mu \mu^{n-m-1} \exp(\tilde{\theta}(\mu)) = 0, \tag{4.9}$$

with asymptotic expansion

$$f(\lambda) \sim \mathrm{O}(\lambda^{-1}), \quad \lambda \to \infty. \tag{4.10}$$

Therefore, for the N-dimensional HMM, the associated solution to the NLS–Toda system hierarchy is given by the initial condition

$$g(\lambda) = \begin{pmatrix} \lambda^N & \lambda^N f(\lambda) \\ 0 & \lambda^{-N} \end{pmatrix}.$$

Theorem 12. *The points, say $W^{(N)}$, in the Sato Grassmannian $\mathrm{Gr}^{(2)}$ corresponding to the solutions of the NLS–Toda hierarchy given by the N-dimensional HMM are*

$$W^{(N)} = \mathbb{C}\{(\lambda^{n+N}, 0), (\lambda^{n+N} f(\lambda), \lambda^{n-N})\}_{n\in\mathbb{N}},$$

where f is the solution of Eq. (4.9) with $b=0$ and asymptotic expansion (4.10).

Observe that they never belong to the Segal–Wilson Grassmannian. If the requirement $b=0$ is removed the above theorem gives the points in the Sato Grassmannian corresponding to general self-similar solutions of the NLS system hierarchy obtained after N consecutive Bäcklund transformations—Toda chain—of the heat hierarchy reduction.

5. Derivative NLS type hierarchies, the Volterra chain and Bäcklund transformations

For the double scaling limit of the HMM one needs to disconsider the odd flows, that is $t_{2n+1} = 0$. It is well known that the semi-infinite Toda chain is replaced by the semi-infinite Volterra chain [9]. As we shall show, this is connected with the derivative NLS system type hierarchies.

However, let us first introduce a slight generalization of the lattice-differential integrable hierarchy appearing in this reduction.

Definition 13. For each $c \in \mathbb{C}$ the generalized derivative NLS system hierarchy, denoted by dNLS(c), for the couple of functions u, v depending on $\boldsymbol{t} = \{t_{2n}\}_{n\geq 0}$ is the following set of compatible equations:

$$\partial_{2n}u = u_{2n+1} + (c-1)\ell_{2n}u, \qquad \partial_{2n}v = -v_{2n+1} - (c-1)\ell_{2n}v, \tag{5.1}$$

where u_{2n+1}, v_{2n+1} and ℓ_{2n} are defined recursively by the relations

$$\begin{aligned} u_{2n+1} &= \partial_2 u_{2n-1} + 2u\ell_{2n} + (c+1)uvu_{2n-1}, \\ v_{2n+1} &= -\partial_2 v_{2n-1} + 2v\ell_{2n} + (c+1)uvv_{2n-1}, \\ \partial_2\ell_{2n} &= (c+1)uv\partial_2\ell_{2n-2} - \left(u\partial_2 v_{2n-1} + v\partial_2 u_{2n-1}\right), \end{aligned}$$

with the initial data $\ell_0 = 1,\ u_1 = u,\ v_1 = v,\ \ell_2 = -uv$.

For $n = 2$ the equations are

$$\begin{aligned} \partial_4 u &= \partial_2^2 u + 2(c-1)uv\partial_2 u + 2cu^2\partial_2 v - c(c+1)u^3v^2, \\ \partial_4 v &= -\partial_2^2 v + 2(c-1)uv\partial_2 v + 2cv^2\partial_2 u + c(c+1)u^2v^3, \end{aligned}$$

a complex version of a system considered in [7], which for $c = 0$ is the system analyzed in [5] and when $c = -1$ is the dNLS equation studied in [17].

If ∂_2^{-1} is a primitive of ∂_2 one constructs the following Miura type transformation:

Proposition 14. *The pair of functions given by*

$$u(c') = u(c)\exp\left(-(c-c')\partial_2^{-1}(u(c)v(c))\right), \tag{5.2}$$

$$v(c') = v(c)\exp\left((c-c')\partial_2^{-1}(u(c)v(c))\right), \tag{5.3}$$

is a solution to the dNLS(c') if $u(c), v(c)$ are solutions of the dNLS(c).

One has

$$R(c) := -u(c)v(c) = R(c') =: R.$$

In the next section we shall prove, by geometrical means, the following

Proposition 15. *Given $\{u^{(n)}, v^{(n)}\}_{n\in\mathbb{Z}}$ satisfying the lattice equations*

$$\partial_2 u^{(n)} = -u^{(n)}(u^{(n+1)}v^{(n+1)} + cu^{(n)}v^{(n)}), \tag{5.4}$$

$$\partial_2 v^{(n+1)} = v^{(n+1)}(cu^{(n+1)}v^{(n+1)} + u^{(n)}v^{(n)}) \tag{5.5}$$

with

$$u^{(n)}v^{(n+1)} = -\exp\left((c-1)\partial_2^{-1}(u^{(n+1)}v^{(n+1)} - u^{(n)}v^{(n)})\right), \tag{5.6}$$

then, if for some particular n the functions $u^{(n)}, v^{(n)}$ are solutions of the dNLS(c) system hierarchy, for any integer m the couple $u^{(m)}, v^{(m)}$ is a solution as well.

Observe that the equation satisfied by $R_n = -u^{(n)}v^{(n)}$ is the Volterra chain

$$\partial_2 R_n = R_n(R_{n+1} - R_{n-1}). \tag{5.7}$$

This lattice gives non-local auto-Bäcklund transformations for the dNLS(c) system hierarchy as follows.

Proposition 16. *The transformation*

$$u^{(n+1)} = \left(u^{(n)}\right)^c \exp\left((c^2-1)\partial_2^{-1}(u^{(n)}v^{(n)})\right)(\partial_2 \ln u^{(n)} + cu^{(n)}v^{(n)}), \tag{5.8}$$

$$v^{(n+1)} = -\left(u^{(n)}\right)^{-c} \exp\left(-(c^2-1)\partial_2^{-1}(u^{(n)}v^{(n)})\right), \tag{5.9}$$

maps a solution $(u^{(n)}, v^{(n)})$ to a new solution $(u^{(n+1)}, v^{(n+1)})$ of the dNLS(c) system hierarchy.

Proof. Eq. (5.4) gives $u^{(n+1)}v^{(n+1)}$ in terms of $u^{(n)}$ and $v^{(n)}$; introducing this in Eq. (5.6) one concludes

$$\left(u^{(n)}\right)^c v^{(n+1)} = -\exp\left((1-c^2)\partial_2^{-1}(u^{(n)}v^{(n)})\right).$$

Using this and Eq. (5.4) one has the desired proof. □

Observe that in the above proof we have used the symmetry $u \to hu$, $v \to h^{-1}v$, for some constant h, of the dNLS(c) system hierarchy.

Only in the cases $c = \pm 1$ do Eqs. (5.8) and (5.9) give local auto-Bäcklund transformations. In general, it is possible to construct a local auto-Bäcklund transformation if one considers in a separate way even and odd sites in the lattice.

Proposition 17. *The transformation defined by*

$$\begin{aligned} u^{(n+2)} = {} & u^{(n)}\left(\partial_2 \ln u^{(n)} + cu^{(n)}v^{(n)}\right)^c \\ & \times \left(\partial_2(\ln(\partial_2 \ln u^{(n)} + cu^{(n)}v^{(n)})) - u^{(n)}v^{(n)}\right), \end{aligned} \tag{5.10}$$

$$v^{(n+2)} = -\left(u^{(n)}\right)^{-1}\left(\partial_2 \ln u^{(n)} + cu^{(n)}v^{(n)}\right)^{-c}, \tag{5.11}$$

gives solutions $u^{(n+2)}, v^{(n+2)}$ from solutions $u^{(n)}, v^{(n)}$ to the dNLS(c) system hierarchy.

Proof. Using Eqs. (5.4) and (5.9) one can verify that the following identity holds:

$$u^{(n)}v^{(n+2)} = -\left(-u^{(n+1)}v^{(n+1)}\right)^{-c}.$$

This implies Eq. (5.11). The Volterra chain (5.7) gives

$$u^{(n+2)}v^{(n+2)} = -\partial_2(\ln(\partial_2 \ln u^{(n)} + cu^{(n)}v^{(n)})) - u^{(n)}v^{(n)},$$

which together with the just deduced Eq. (5.11) ensures the truth of Eq. (5.10). □

In general we have two separate families $\{u^{(2n)}, v^{(2n)}\}$ and $\{u^{(2n+1)}, v^{(2n+1)}\}$ satisfying the same lattice.

As examples we consider three particular values of c, namely $c = 1, -1, 0$. Recall that when $c = \pm 1$ then Eqs. (5.8) and (5.9) are local. Thus, we consider the induced transformation. When $c = 1$ one has

$$u^{(n+1)} = \partial_2 u^{(n)} + \frac{1}{v^{(n)}\left(u^{(n)}\right)^2}, \qquad v^{(n+1)} = -1/u^{(n)}.$$

For $c = -1$ we have

$$u^{(n+1)} = \frac{\partial_2 u^{(n)}}{\left(u^{(n)}\right)^2} - v^{(n)}, \qquad v^{(n+1)} = -u^{(n)},$$

the lattice considered in [27,24,1], giving auto-Bäcklund transformations for the dNLS system hierarchy.

When $c = 0$ Eqs. (5.8) and (5.9) are non-local, hence we study Eqs. (5.10) and (5.11),

$$\begin{aligned} u^{(n+2)} &= \partial_2^2 u^{(n)}/\partial_2 u^{(n)} - \partial_2 u^{(n)} - v^{(n)}\left(u^{(n)}\right)^2, \\ v^{(n+2)} &= -1/u^{(n)}, \end{aligned}$$

a Bäcklund transformation, that with the notation $u^{(n)} = \exp(\phi_n)$ reads as a modification of the Toda chain

$$\partial_2^2 \phi_n = \left(\exp(\phi_{n+2} - \phi_n) - \exp(\phi_n - \phi_{n-2})\right)\partial_2 \phi_n. \tag{5.12}$$

We see that there are two families $\{u^{(2n)}, v^{(2n)}\}_{n\in\mathbb{Z}}$ and $\{u^{(2n+1)}, v^{(2n+1)}\}_{n\in\mathbb{Z}}$ of solutions of (5.12) connected through

$$\partial_2 u^{(n)} = -u^{(n)}u^{(n+1)}v^{(n+1)}.$$

6. The periodic flag manifold and the dNLS(c) system hierarchy

The dNLS(c) system hierarchy can be understood through a factorization problem associated to a particular solution of the modified classical Yang–Baxter equation [10].

In this section we are going to generalize that construction in order to obtain the Bäcklund transformations of the previous section. The Sato periodic flag manifold $\mathrm{Fl}^{(2)}$ [21] and certain line bundles over it replace the Sato Grassmannian.

The Coxeter automorphism [14] generated by the adjoint action of $C = \exp(i\pi H/2)$ defines the Lie subalgebra

$$L(\mathfrak{sl}(2,\mathbb{C}),C) = \{X \in L\mathfrak{sl}(2,\mathbb{C}) : X(-\lambda) = \mathrm{Ad}C\ X(\lambda)\}.$$

This is the set of loops of the form $\lambda b(\lambda^2)E + a(\lambda^2)H + \lambda^{-1}c(\lambda^2)F$. This subalgebra is isomorphic to $L\mathfrak{sl}(2,\mathbb{C})$; in fact is the principal realization of the affine Lie algebra of the type $A_1^{(1)}$ [14]. Its root subsystem $\Delta_C = \{2n\delta, 2n\delta \pm \alpha_1\}_{n\in\mathbb{Z}}$, is invariant under the action of the translation group of $L\mathfrak{sl}(2,\mathbb{C})$. The corresponding Lie group is

$$L(\mathrm{SL}(2,\mathbb{C}),C) = \{g \in L\mathrm{SL}(2,\mathbb{C}) : Cg(-\lambda) = g(\lambda)C\},$$

its elements being of the form

$$\begin{pmatrix} a(\lambda^2) & \lambda b(\lambda^2) \\ \lambda^{-1}c(\lambda^2) & d(\lambda^2) \end{pmatrix},$$

where $ad - bc = 1$ [26].

Consider the following reduction of the vacuum wave function introduced in (4.1):

$$\psi^{(\mathrm{even})}(t,\lambda) := \exp\left(V^{(\mathrm{even})}(t,\lambda)\ H/2\right) \cdot g(\lambda),$$

with

$$V^{(\mathrm{even})}(t,\lambda) := \sum_{n\geq 0} \lambda^{2n} t_{2n}$$

and $g \in L(\mathrm{SL}(2,\mathbb{C}),C)$.

The left multiplication by the translation element T is not well defined for this reduction. Nevertheless, observe that in the non-reduced case the transformation $g^{(n)} \mapsto g^{(n+1)} = T^{-1} \cdot g^{(n)} \cdot a$ with $a \in L^+\mathrm{SL}(2,\mathbb{C})$ gives the same results as those stated in the previous section. This allows us to construct an action of the translation group over our reduced vacuum wave function, i.e. preserving the reduction; one only needs to choose $a = r_1\tilde{a}$ where $r_1 := \exp(E)\exp(-F)\exp(E)$ and $\tilde{a} \in L(\mathrm{SL}(2,\mathbb{C}),C) \cap L^+\mathrm{SL}(2,\mathbb{C})$.

Proposition 18. *The action of the translation group given by*

$$g \mapsto T^{-1} \cdot g \cdot r_1, \quad g \in L\mathrm{SL}(2,\mathbb{C})$$

reduces to an action over the principal subgroup $L(\mathrm{SL}(2,\mathbb{C}),C)$

So we define

Definition 19. The vacuum wave functions are defined by

$$\psi^{(\mathrm{even},n)} := T^{-n} \cdot \psi^{(\mathrm{even})} \cdot r_1^n. \tag{6.1}$$

Note that $\psi^{(\text{even},2n)} = -T^{-2n}\cdot\psi^{(\text{even})}$, in fact T^{-2} is the generator, in the principal realization, of the translation group of $A_1^{(1)}$, see Section 4.

We now modify the induced Birkhoff factorization in $L(\mathrm{SL}(2,\mathbb{C}),C)$ by considering a classical r-matrix [25]. The resolution of the identity $\mathrm{id} = P_+ + P_0 + P_-$ associated with the splitting

$$L(\mathfrak{sl}(2,\mathbb{C}),C) = L_1^+(\mathfrak{sl}(2,\mathbb{C}),C)\oplus\mathbb{C}H\oplus L_1^-(\mathfrak{sl}(2,\mathbb{C}),C)\,,$$

where L_1^+ stands for those loops with holomorphic extension to the interior of the circle normalized by 0 at $\lambda=0$, and L_1^- for those with analytical extension to the exterior of the circle and normalized by 0 at $\lambda=\infty$, gives a classical r-matrix $R := P_+ + cP_0 - P_-$ for any $c\in\mathbb{C}$. The exponentiation, say $\mathrm{R}_\pm$, of the endomorphisms $R_\pm := (R\pm 1)/2 = \pm P_\pm + (c\pm 1)/2\;P_0$ to the Lie group $L(\mathrm{SL}(2,\mathbb{C}),C)$ gives a natural extension of the Birkhoff factorization, see [10,25]. The factorization to consider is

$$\psi^{(\text{even},n)} = \left(\psi_-^{(n)}\right)^{-1}\cdot\psi_+^{(n)}$$

with $\psi_\pm^{(n)}(t)\in\mathrm{R}_\pm L(\mathrm{SL}(2,\mathbb{C}),C)$ satisfying the Cayley condition $\Theta(\psi_+\cdot K_+) = \psi_-\cdot K_-$, see [10,25]. The solution to this factorization can be expressed [10] in terms of functions $u^{(n)}, v^{(n)}$, solutions of Eqs. (5.1). The procedure is analogous to the one exposed in the previous section.

Instead of Eqs. (4.3) and (4.5) we have the formulae

$$\chi^{(n)} = d\psi_+^{(n)}\cdot\left(\psi_+^{(n)}\right)^{-1} = R_-\,\mathrm{Ad}\,\psi_-^{(n)}\mathrm{d}V^{(\text{even})}\,H/2\,, \tag{6.2}$$

$$\mathcal{T}_+^{(n)} := \psi_+^{(n+1)}\cdot r_1^{-1}\cdot\left(\psi_+^{(n)}\right)^{-1} = \psi_-^{(n+1)}\cdot T^{-1}\cdot\left(\psi_-^{(n)}\right)^{-1}. \tag{6.3}$$

The lattice-differential integrable hierarchy can be formulated as in Eqs. (4.4) and (4.6). The parametrization of $\psi_-^{(n)}$ in terms of the functions $u^{(n)}, v^{(n)}$ gives the following expressions:

$$\chi^{(n)} = \sum_{m\geq 0} L_{2m}^{(n)}\,dt_{2m}\,,$$

with

$$L_{2m}^{(n)} := \sum_{j=0}^{2m-1}\lambda^{2m-j}Q_j^{(n)} + \tfrac{1}{2}(c+1)Q_{2m}^{(n)}\,,$$

where

$$Q_{2j}^{(n)} := \ell_{2j}^{(n)}H,\qquad Q_{2j+1}^{(n)} := u_{2j+1}^{(n)}E + v_{2j+1}^{(n)}F,$$

and

$$\mathcal{T}_+^{(n)} = \exp\left(-\tfrac{1}{2}(c-1)\partial_2^{-1}(u^{(n+1)}v^{(n+1)} - u^{(n)}v^{(n)})\right)\begin{pmatrix}\lambda & u^{(n)}\\ v^{(n+1)} & 0\end{pmatrix}.$$

When $c = 1$ the factorization is the one induced by the Birkhoff factorization in $L\mathrm{SL}(2,\mathbb{C})$. The dNLS(1) system hierarchy can be obtained from the NLS system hierarchy by reduction, just take the odd times equal to zero, $t_{2n+1} = 0$, and the initial condition $g \in L(\mathrm{SL}(2,\mathbb{C}), C)$, then identify $u = p$, $v = q$. The action of the translation group when reduced is the one given in (6.1). In this case the moduli space will be the following homogeneous space:

$$L(\mathrm{SL}(2,\mathbb{C}), C)/L^+(\mathrm{SL}(2,\mathbb{C}), C).$$

Because of the principal isomorphism the following identifications hold [26]:

$$L(\mathrm{SL}(2,\mathbb{C}), C) \cong L\mathrm{SL}(2,\mathbb{C}),$$
$$L^+(\mathrm{SL}(2,\mathbb{C}), C) \cong B^+\mathrm{SL}(2,\mathbb{C}),$$

where $B^+\mathrm{SL}(2,\mathbb{C})$ is the set of maps in $L^+\mathrm{SL}(2,\mathbb{C})$ such that its holomorphic extension to the interior of S^1 when evaluated at $\lambda = 0$ is an upper triangular matrix. Therefore, the moduli space is isomorphic to the periodic flag manifold $\mathrm{Fl}^{(2)}$ [21].

The periodic flag manifold $\mathrm{Fl}^{(2)}$ is the set of couples of subspaces, say $V, W \in \mathrm{Gr}^{(2)}$, such that the periodicity condition $\lambda W \subset V \subset W$ holds and $\dim W/V = 1$ [21]. Given a initial condition

$$g(\lambda) = \begin{pmatrix} \varphi_1(\lambda^2) & \lambda\tilde{\varphi}_1(\lambda^2) \\ \lambda^{-1}\varphi_2(\lambda^2) & \tilde{\varphi}_2(\lambda^2) \end{pmatrix} \in L(\mathrm{SL}(2,\mathbb{C}), C)$$

and using the notation $\Phi = (\varphi_1, \varphi_2)$ and $\tilde{\Phi} = (\tilde{\varphi}_1, \tilde{\varphi}_2)$ the associated point in $\mathrm{Fl}^{(2)}$ is (V, W) with

$$V = \mathbb{C}\{\Phi, \lambda^n\Phi, \lambda^n\tilde{\Phi}\}_{n>0}, \qquad W = \mathbb{C}\{\lambda^n\Phi, \lambda^n\tilde{\Phi}\}_{n\geq 0}.$$

The initial conditions associated with the N-dimensional $t_{2n+1} = 0$ reduction of HMM are

$$g = \begin{pmatrix} -\lambda^{2n} & -\lambda^{2n} f(\lambda) \\ 0 & -\lambda^{-2n} \end{pmatrix}$$

for $N = 2n$, and

$$g = \begin{pmatrix} \lambda^{2n+1} f(\lambda) & \lambda^{2n+1} \\ -\lambda^{-2n-1} & 0 \end{pmatrix}$$

for $N = 2n + 1$. Here $f(\lambda) =: \lambda F(\lambda^2)$ is an odd function in λ, a solution of (4.9) with $b = 0$ and $\theta(\lambda) = \sum \theta_{2n}\lambda^{2n}$ and having the asymptotic expansion (4.10).

Theorem 20. *The solutions to the dNLS(1)–Volterra system hierarchy given by the $t_{2n+1} = 0$ reduction of the HMM corresponds to the following points in the Sato periodic flag manifold* $\mathrm{Fl}^{(2)}$.

(i) In the $2m$ *dimensional case*

$$V = \mathbb{C}\{(\lambda^{n+m}, 0), (\lambda^{n+m+1}F(\lambda), \lambda^{n-m+1})\}_{n\geq 0},$$
$$W = \mathbb{C}\{(\lambda^{n+m}, 0), (\lambda^{n+m}F(\lambda), \lambda^{n-m})\}_{n\geq 0}.$$

(ii) For the $2m+1$ *case*

$$V = \mathbb{C}\{(\lambda^{n+m+1}, 0), (\lambda^{n+m+1}F(\lambda), \lambda^{n-m})\}_{n\geq 0},$$
$$W = \mathbb{C}\{(\lambda^{n+m}, 0), (\lambda^{n+m+1}F(\lambda), \lambda^{n-m})\}_{n\geq 0}.$$

The reduced string equation is no longer connected with a symmetry of the integrable hierarchy, as the translational and scaling symmetries preserve the dNLS(1)-reduction of NLS and the Galilean one does not. We are looking for reduced solutions, i.e. solutions of the dNLS(1) system hierarchy which remain fixed under the action of the vector field X of the NLS system hierarchy, and only when $a = 0$ can be understood as a self-similarity condition within the dNLS system hierarchy. Again, the departure point is a solution of the heat hierarchy, $v = 0$ and $\partial_{2n}u = \partial_2^n u$.

Observe that if the condition $b = 0$ is removed then the above theorem gives the points corresponding to general self-similar solutions obtained from the heat hierarchy through a chain of consecutive Bäcklund transformations. None of these points belong to the Segal–Wilson periodic flag manifold.

When $c \neq 1$ the moduli space is not $\mathrm{Fl}^{(2)}$ but a line bundle over it. This follows from the factorization problem, ψ_- takes its values in $L^-(\mathrm{SL}(2,\mathbb{C}), C)$, so that the moduli space is $L(\mathrm{SL}(2,\mathbb{C}), C)/L_1^+(\mathrm{SL}(2,\mathbb{C}), C)$. Recalling that $L_1^+(\mathrm{SL}(2,\mathbb{C}), C) \cong N^+\mathrm{SL}(2,\mathbb{C})$—loops in $L^+\mathrm{SL}(2,\mathbb{C})$ such that its holomorphic extension to the interior of S^1 when evaluated at $\lambda = 0$ is strictly upper triangular—one easily concludes that this moduli space is a line bundle over the periodic flag manifold.

7. Miura type map between the NLS and dNLS(c) hierarchies

Given a solution to the dNLS(c) system hierarchy one can obtain through a Miura type transformation a solution to the NLS system hierarchy, see for example [10].

Proposition 21. *The set of functions* $\{p^{(n)}, q^{(n)}\}_{n\in\mathbb{Z}}$ *defined by*

$$p^{(n)} = u^{(n)} \exp\left(-(c-1)\partial_2^{-1}\left(u^{(n)}v^{(n)}\right)\right), \tag{7.1}$$
$$q^{(n)} = \left(-\partial_2 v^{(n)} + cu^{(n)}\left(v^{(n)}\right)^2\right) \exp\left((c-1)\partial_2^{-1}\left(u^{(n)}v^{(n)}\right)\right), \tag{7.2}$$

are solutions of the NLS system hierarchy if $u^{(n)}, v^{(n)}$ *are solutions of the dNLS(c) system hierarchy.*

For the dNLS(c) we are dealing with the principal realization of an $A_1^{(1)}$ type affine Lie algebra. We can construct the NLS system hierarchy within this principal realization

$(t_n \mapsto t_{2n})$ by modifying the induced Birkhoff factorization. In this approach the NLS–Toda system hierarchy appears associated with the generator T^{-2}. A moment of thought is enough to realize that the geometrical construction of the dNLS(c) system hierarchy can be done in the loop group $L\mathrm{SL}(2,\mathbb{C})$ and inducing the classical r-matrix introduced previously. We choose the principal realization because of the absence of square roots in the spectral parameter.

The periodic flag manifold $\mathrm{Fl}^{(2)}$ is a $\mathbb{C}P^1$ bundle over the Grassmannian $\mathrm{Gr}^{(2)}$, with a projection map, say π. The Miura type map between NLS and dNLS(1) can be interpreted, in the spirit of [27], as this projection map. Moreover, one has the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Fl}^{(2)} & \longrightarrow & \mathrm{NLS} \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle \text{Miura type map}} \\ \mathrm{Gr}^{(2)} & \longrightarrow & \mathrm{dNLS}(1). \end{array}$$

For $c \neq 1$, the periodic flag manifold is replaced by a line bundle over it, say $\mathcal{L}$. The projection map ϖ to $\mathrm{Fl}^{(2)}$ gives the Miura transformation defined by the equations (5.2) and (5.3) between dNLS(c) and dNLS(1). Now, the commutative diagram is

$$\begin{array}{ccc} \mathcal{L} & \longrightarrow & \mathrm{dNLS(c)} \\ {\scriptstyle\varpi}\big\downarrow & & \big\downarrow{\scriptstyle \text{Miura type map: Eqs. (5.2), (5.3) } c\to 1} \\ \mathrm{Fl}^{(2)} & \longrightarrow & \mathrm{dNLS}(1) \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle \text{Miura type map: Eqs. (7.1), (7.2), } c=1} \\ \mathrm{Gr}^{(2)} & \longrightarrow & \mathrm{NLS}. \end{array}$$

The lattice associated to the NLS system hierarchy and the generator T^{-1}—in the principal realization—can be shown to be as follows.

Proposition 22. *Suppose a set $\{p^{(n)}, q^{(n)}\}_{n\in\mathbb{Z}}$ satisfying*

$$\begin{aligned} \partial_2 p^{(n)} &= p^{(n+1)} - \left(p^{(n)}\right)^2 q^{(n+1)}, \\ \partial_2 q^{(n+1)} &= -q^{(n)} + p^{(n)}\left(q^{(n+1)}\right)^2, \\ q^{(n+2)} &= -1/p^{(n)}. \end{aligned}$$

Suppose also that for some n the functions $p^{(n)}, q^{(n)}$ are solutions of the NLS system hierarchy. Then, all of them are solutions.

Eqs. (4.4) and (4.6) represent this lattice-differential system but now the 1-form $\chi^{(n)}$ is the one defined in (4.7) after the map $\lambda^n E \mapsto \lambda^{2n+1}E$, $\lambda^n H \mapsto \lambda^{2n}H$, $\lambda^n F \mapsto \lambda^{2n-1}F$ and instead of $\mathcal{T}_+^{(n)}$ one uses

$$\Theta_+^{(n)} = \begin{pmatrix} \lambda + \lambda^{-1}p^{(n)}q^{(n+1)} & p^{(n)} \\ \lambda^{-2}q^{(n+1)} & \lambda^{-1} \end{pmatrix}.$$

The families $\{p^{(2n)}, q^{(2n)}\}_{n\in\mathbb{Z}}$ and $\{p^{(2n+1)}, q^{(2n+1)}\}_{n\in\mathbb{Z}}$ are solutions of the NLS–Toda system hierarchy. The connection between both is

$$p^{(n+1)} = \partial_2 p^{(n)} - \left(p^{(n)}\right)^2 / p^{(n-1)}. \tag{7.3}$$

Observe that $\mathcal{T}_+^{(n+1)} = \Theta_+^{(n+1)} \cdot \Theta_+^{(n)}$, so this lattice can be considered as a "square root" of the NLS–Toda system hierarchy.

The *mysterious* square root of the Toda chain or of the Weyl action can be easily understood once the loop group $L\mathrm{SL}(2,\mathbb{C})$ is embedded as a subgroup in a larger one, for example $L\mathrm{SL}(3,\mathbb{C})$. For this case, $A_2^{(1)}$, the translational subgroup of the affine Weyl group is a 2-dimensional Abelian group with generators T_1, T_2 [14]. The integrable system associated with the Birkhoff factorization in this loop group will be a generalized NLS system hierarchy [8].

If we consider the reduction to the subgroup $LS(\mathrm{GL}(2,\mathbb{C})\times\mathbb{C})$ we have that the action of the translational group by right multiplication of $T_1^{-1} = \mathrm{diag}(\lambda, \lambda^{-1}, 1)$ and $T_2^{-1} = \mathrm{diag}(1, \lambda, \lambda^{-1})$ preserves the reduction. To each $g = (h, 1/\det h) \in LS(\mathrm{GL}(2,\mathbb{C})\times\mathbb{C})$, $h \in L\mathrm{GL}(2,\mathbb{C})$, we associate $\tilde{g} := (1/\sqrt{\det h}) \cdot h \in L\mathrm{SL}(2,\mathbb{C})$. Moreover, with this map the reduction gives the NLS system hierarchy and the action of $\tilde{T}_1 = T$ is the one giving the Toda chain. For T_2 we have

$$\widetilde{T_2 \cdot g} = T^{-1/2} \cdot \tilde{g}.$$

This implies that the square root is a consequence of the reduction $A_2^{(1)} \to A_1^{(1)}$, and models the reduced action of the T_2.

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